

Ergodic Theory - Week 1

Course Instructor: Florian K. Richter
Teaching assistant: Konstantinos Tsinas

1 Measure Preserving Systems

- P1.** Let $(X, \mathcal{A}, \mu)$ be a probability space, and let $T : X \rightarrow X$ be a measurable map. Show that T preserves μ if and only if

$$\int f \circ T \, d\mu = \int f \, d\mu \quad (1)$$

holds for any $f \in L^1(X)$.

Optional: If X is a Polish space (Hausdorff completely metrizable topological space) and μ is a Borel probability measure on X show that it suffices to check (1) for any $f \in C(X)$.

Hint: Use the fact that any finite Borel measure on a Polish space is regular, namely for any measurable set A and any $\varepsilon > 0$ there exist open U and compact K such that $K \subseteq A \subseteq U$ and $\mu(U \setminus K) < \varepsilon$.

- P2.** Let $(X_1, \mathcal{A}_1, \mu_1, T_1)$ and $(X_2, \mathcal{A}_2, \mu_2, T_2)$ be measure preserving systems. Prove that the product system $(X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mu_1 \otimes \mu_2, T_1 \times T_2)$ is also measure-preserving.
- P3.** We consider the torus with the Borel σ -algebra and the Lebesgue measure.

- (a) Show that for any $a \in \mathbb{R}$, the map $Tx = x + a \pmod{1}$ preserves the Lebesgue measure.
- (b) For each $p \in \mathbb{N}$ we define the map $T_p x = px \pmod{1}$ for all $x \in [0, 1)$. Show that the transformation T_p preserves the Lebesgue measure.

- P4.** Let $(X, \mathcal{A}, \mu)$ be a probability space and let $T : X \rightarrow X$ be an invertible measure preserving transformation (with respect to μ). Now, at any moment, instead of moving forward by T (that is, instead of looking at the map $x \rightarrow Tx$), we flip a fair coin to decide whether we will use T or T^{-1} .

The goal is to describe the random system described above by means of a measure-preserving system. In particular, we want to find a map R such that given a point x and a sequence of coin tosses ω , we would have that $R(\omega, x)$ would produce the same result as the procedure above.

- (a) Find a probability space $(Y, \mathcal{B}, \nu)$ and a measure preserving map S that models the sequence of coin tosses.
 - (b) Consider the product system $(X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \nu)$. Define a measure-preserving map R on this product space that models the original random system.
- P5.** A set $R \subseteq \mathbb{Z}$ is a set of recurrence if for every measure-preserving system $(X, \mathcal{A}, \mu, T)$ and for all $A \in \mathcal{A}$ with $\mu(A) > 0$ we have $\mu(A \cap T^{-n}A) > 0$ for some $n \in R \setminus \{0\}$.
- (a) Show that $2\mathbb{N}$ is a set of recurrence but $2\mathbb{N} + 1$ is not.
 - (b) Show that sets of recurrence possess the Ramsey property: if $R = R_1 \cup \dots \cup R_k$ is a set of recurrence, then one of the sets $R_1, \dots, R_k$ is also a set of recurrence.

(c) Show that if $R \subseteq \mathbb{Z}$ is a set of recurrence, then so is $R \cap m\mathbb{Z}$ for every $m \in \mathbb{Z}$.

P6. Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving system and let $A \in \mathcal{A}$ be a set of positive measure. Define its set of return times

$$R(A) = \{n \in \mathbb{N} : \mu(A \cap T^{-n}A) > 0\}.$$

Poincare's recurrence theorem asserts that $R(A)$ is non-empty. In this exercise, we want to show that $R(A)$ is also large in some appropriate sense.

- (a) Show that $R(A)$ intersects any difference set: if $E \subset \mathbb{N}$ is infinite, then $R(A) \cap (E - E) \neq \emptyset$. Here, $E - E = \{n \in \mathbb{N} : n = a - b \text{ for some } a, b \in E, a > b\}$.
- (b) Show that the set $R(A)$ has bounded gaps, that is there exists $k \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ we have $R(A) \cap \{m, m + 1, \dots, m + k\} \neq \emptyset$.